On the existence of self-similar solutions of the equations governing unsteady flow through a porous medium

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In this paper the conditions for the existence of self-similar solutions of the equations governing unsteady flows through a porous medium are presented and discussed. The first two sections deal with the case of non-Newtonian fluids of power-law behavior; the third section analyzes the case of non-Darcy gas flows. The boundary and initial conditions occurring currently in a large class of fluid mechanics problems, of practical interest in engineering, are considered.

Keywords: unsteady flow through porous media; non-Newtonian fluids; non-Darcy gas flow; self-similar solutions

Introduction

The flow through a porous medium is a topic of special interest in many applications of engineering science. Recently, with the increasing interest in the production of heavy and waxy crude oils, it has become essential to have an adequate understanding of the rheological effects of these non-Newtonian fluids on the flow through a porous medium. As a result, a great deal of interest has focused on this matter. The flow of non-Newtonian displacing fluids of power-law behavior is a topic of great interest at this time in oil reservoir engineering, due to the possibility of improving the oil recovery efficiency from water flooding projects. As the experimental and theoretical evidence has shown, certain non-Newtonian displacing fluids, in particular pseudoplastic behavior, may minimize the effects which tend to destabilize the interface movement separating the displacing and displaced fluids. For example, the polymer solutions and emulsions of oil in water appear to respond favorably to the elimination of viscous fingering effect in oil displacement mechanism. In a certain range of shear rate variation these fluids are non-Newtonian, having a pseudoplastic behavior, in which the apparent viscosity is a decreasing function of increasing shear rate. Therefore, in the flow of these fluids through a porous medium we are faced with a problem in which the rheological effects on the flow are flow rate dependent. The practical and theoretical interest in knowing of these effects is quite justified.

Another class of flows through a porous medium is non-Darcy gas flows under high pressure, in which case the equations governing unsteady flow are nonlinear. This class of flows appears as a result of Darcy's law violation at high velocity. In this situation, the nonlinear inertial effects of convective accelerations and decelerations due to flow through varying cross-sectional areas in the pore space, along with compressibility effect, must be considered.

An efficient approach to deal with the problems mentioned above is the finding of self-similar solutions of the equations governing non-Newtonian flows of power-law fluids and non-Darcy gas. The interest in such solutions appears as a result of the fact that the solution of a partial differential equation, for example, describing the flow in an oil reservoir, may be reduced in certain cases of practical interest to the solution of an ordinary differential equation. In this way it could be possible, as we will see further on, to obtain an exact solution, sometimes even in a closed form, by means of an elementary approach. It is always worthwhile to look for self-similar solutions before we use a numerical approach to solve the nonlinear equations governing unsteady flows through porous media.

A great number of publications have been devoted to the use of self-similar solutions in the solving of fluid mechanics problems related to the flow through a porous medium. It is outside the scope of this study to review these papers here. However, it should be pointed out that the publications are concerned with Newtonian fluids. As far as the author is aware, the case of non-Newtonian fluids of power-law behavior has not been presented in the literature. Our objective in this paper is to show the existence of self-similar solutions for some problems of fluid mechanics of interest at this time in oil reservoir engineering. The paper is organized as follows: in the first two sections we deal specifically with unsteady flow of power-law fluids, showing the existence of self-similar solutions for certain boundary conditions occurring in practice. In the third section we focus on a more general case which includes non-Darcy gas flows as well.

Unsteady flow of power-law fluids

One-dimensional problem

As we previously showed,[1] the unsteady flow equations for a slightly compressible fluid of power-law behavior may be written

$$-\frac{\partial p}{\partial x}=\frac{\mu_{\text{eff}}}{k}v^{n} \tag{1}$$

$$-\beta\phi\frac{\partial p}{\partial t}=\frac{\partial v}{\partial x} \tag{2}$$

where

$$\frac{k}{\mu_{\text{eff}}}=\frac{1}{2H}\left(\frac{n\phi}{1+3n}\right)^{n}\left(\frac{8k}{\phi}\right)^{(n+1)/2} \tag{3}$$

Notations are shown in the notation given at the beginning of the paper.

In the basic Equation 1, the inertial effects have been disregarded. This assumption seems to be reasonably correct when we are dealing with the flow of viscous fluids through a porous medium. Equation 1 is a modified Darcy's law including the rheological effects of power-law fluids.

As shown by Pascal and Pascal,[2] using the transformation

$$\eta = xt^{-n/(1+n)} \tag{4}$$

which is the self-similar variable for Equations 1 and 2, the following system of ordinary differential equations is obtained

$$v^n = -\frac{k}{\mu_{\text{eff}}} t^{-n/(1+n)} \frac{dp}{d\eta} \tag{5}$$

and

$$\frac{dv}{d\eta} = +\frac{n\beta\phi}{1+n} \eta t^{-1/(1+n)} \frac{dp}{d\eta} \tag{6}$$

From Equations 1, 2, and 4 one obtains

$$\frac{d^2p}{d\eta^2} + \frac{n^2a^2}{1+n}\eta\left(\frac{dp}{d\eta}\right)^{(2n-1)/n} = 0 \tag{7}$$

in which

$$a^2 = \left(\frac{\mu_{\text{eff}}}{k}\right)^{1/n}(\beta\phi) \tag{8}$$

Equation 7 determines the pressure distribution in the flow system provided that appropriate boundary conditions on function $p(\eta)$ are specified. Once $p(\eta)$ is known, the velocity distribution, expressed by the function $v(\eta)$, may also be known from Equation 5.

Assuming a flow system of infinite extent, depleted at a constant pressure at the outface flow, then for this case the appropriate boundary conditions will be

$$\begin{aligned} &\eta = 0 \qquad p = p_w = \text{constant} \\ &\eta = \infty \qquad p = p_k = \text{constant and } v = 0;\ p_k > p_w \end{aligned} \tag{9}$$

From a practical point of view, the interest is to determine $v_0(t)$ rather than function $v(\eta)$, where $v_0(t) = v(0)$ represents the velocity variation with time at the outface flow, such that a constant pressure of production may be maintained there. In this case, it is convenient to express the pressure and velocity distributions by the following functions:

$$p = p_w f(\eta) \tag{10}$$

$$v = v_0(t)\Phi(\eta) \tag{11}$$

Introducing Equations 10 and 11 into Equations 5 and 6, we have the system of nonlinear equations for $f(\eta)$ and $\Phi(\eta)$:

$$\Phi(\eta) = -\left(\frac{kp_w}{\mu_{\text{eff}}}\right)^{1/n} \frac{t^{-1/(1+n)}}{v_0(t)} \left(\frac{df}{d\eta}\right)^{1/n} \tag{12}$$

and

$$\frac{d\Phi}{d\eta} = +\frac{n\beta\phi p_w}{1+n} \frac{\eta t^{-1/(1+n)}}{v_0(t)} \frac{df}{d\eta} \tag{13}$$

in which, from 9, 10, and 11, the following boundary conditions arise

$$\begin{aligned} &\eta = 0 \qquad f(0) = 1 \quad \text{and} \quad \Phi(0) = 1 \\ &\eta = \infty \qquad f(\infty) = \frac{p_k}{p_w} \quad \text{and} \quad \Phi(\infty) = 0 \end{aligned} \tag{14}$$

Equations 12 and 13 lead to an equation identical with Equation 7 in function $f(\eta)$, provided that an appropriate expression for $v_0(t)$ is found

$$\frac{d^2f}{d\eta^2} + \frac{n^2a^2}{1+n}\eta\left(\frac{df}{d\eta}\right)^{(2n-1)/n} = 0 \tag{15}$$

while from 12, taking into account that $\Phi(0) = 1$, one obtains

$$v_0(t) = -\left(\frac{kp_w}{\mu_{\text{eff}}}\right)^{1/n} t^{-1/(1+n)} \left(\frac{df}{d\eta}\right)^{1/n}_{\eta=0} \tag{16}$$

Previous relations 11, 12 and 13 yield

$$\Phi(\eta) = \left[\frac{df/d\eta}{df/d\eta|_{\eta=0}}\right]^{1/n} \tag{17}$$

Since $(df/d\eta)^{1/n}_{\eta=0} = \text{constant}$, then from 12, 13, and 16 it is evident that function $v_0(t)$ is of the form

$$v_0(t) = At^{-1/(1+n)}; \qquad A = -\left(\frac{kp_w}{\mu_{\text{eff}}}\right)^{1/n}\left(\frac{df}{d\eta}\right)^{1/n}_{\eta=0} \tag{18}$$

We now turn our attention to the determination of $(df/d\eta)^{1/n}_{\eta=0}$ occurring in 18. For this purpose, the solution of Equation 15, satisfying conditions specified in 14, is required. Equation 15 may be integrated twice to yield

$$\frac{df}{d\eta} = \left[C_1 - \frac{na^2}{2}\frac{1-n}{1+n}\eta^2\right]^{n/(1-n)} \tag{19}$$

Notation

F	Cross-sectional area
$G_0(t)$	Cumulative production
h	Oil reservoir thickness
H	Consistency index (coefficient in power-law equation)
k	Permeability
n	Power-law exponent
p	Pressure distribution in the flow system
p_w	Pressure at the outface flow
p_k	Pressure at the initial moment, $t = 0$
Q_w	Volumetric flow rate
R	Radial distance
R_w	Well radius
t	Time
v	Velocity distribution in the flow system
$v_0(t)$	Velocity variation in time at the outface flow

Greek letters

$\bar{\beta}$	Inertial flow resistance coefficient
β	Compressibility coefficient for slightly compressible fluid
ϕ	Porosity
γ	Density
$\dot{\gamma}$	Shear rate
μ_a	Apparent viscosity for a power-law fluid
μ_{eff}	Effective viscosity for a power-law fluid flowing through a porous medium
$q = \rho v$	Mass velocity
ρ	Specific mass
ρ_k	Specific mass corresponding to the initial reservoir pressure, $t = 0$

whereas

$$f(\eta)=1+\int_0^{\eta}\left[C_1-\frac{na^2}{2}\frac{1-n}{1+n}\eta^2\right]^{n/(1-n)}\mathrm{d}\eta \tag{20}$$

For $\eta=0$ from 19 we have $(\mathrm{d}f/\mathrm{d}\eta)^{1/n}_{\eta=0}=C_1=\text{constant}$. To determine C_1 one can use the condition shown in 14; i.e., $\eta=\infty$, if $f(\infty)=p_k/p_w$, for $n>1$. The case $n<1$, corresponding to a non-Newtonian fluid of pseudoplastic behavior, requires a finite interval of variation of the variable η; i.e., $0<\eta<\eta_1$, in the formula 20. Therefore, for $n<1$, we are led to the formulation of a moving boundary problem, in which the location of a time-dependent boundary is determined by the relation 4

$$l(t)=\eta_1 t^{n/(1+n)} \quad \text{with } \eta_1=\text{constant} \tag{21}$$

In a previous paper, Pascal and Pascal,[2] we have shown how the solution of Equation 15 can be analytically obtained from a certain formulation of the moving boundary problem. The interested reader is referred to Pascal and Pascal[2] for more detailed coverage of this subject.

As given in Pascal and Pascal,[2] solutions of the nonlinear Equations 1 and 2 for $n<1$ are expressed as

$$p(\eta)-p_w=B^{n/(1-n)}\eta_1^{(1+n)/(1-n)}J_n\left(\frac{\eta}{\eta_1}\right); \qquad 0<\frac{\eta}{\eta_1}<1 \tag{22}$$

and

$$v(x,t)=\left(\frac{k}{\mu_{\mathrm{eff}}}\right)^{1/n}t^{-1/(1+n)}\left[B\left(\eta_1^2-\frac{x^2}{t^{2n/(1+n)}}\right)\right]^{1/(1-n)} \tag{23}$$

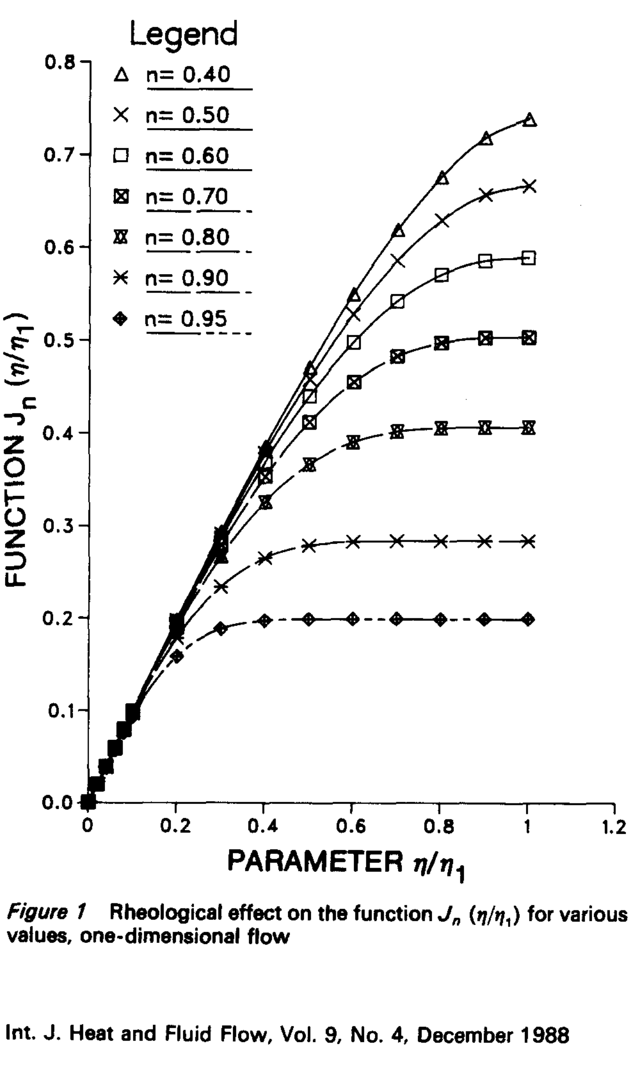

Figure 1 Rheological effect on the function J_n (η/η_1) for various n values, one-dimensional flow

in which

$$\frac{\eta}{\eta_1}=\frac{x}{l(t)}; \qquad \eta_1=B^{-n/(1+n)}\left[\frac{\Delta p}{L_n}\right]^{(1-n)/(1+n)}; \qquad B=\frac{na^2}{2}\frac{1-n}{1+n} \tag{24}$$

and

$$J_n\left(\frac{\eta}{\eta_1}\right)=\int_0^{\eta/\eta_1}(1-\xi^2)^{n/(1-n)}\,\mathrm{d}\xi; \qquad \Delta p=p_k-p_w \tag{25}$$

with $L_n=J_n(1)$ and $l(t)$ given by the relation 21. Once $l(t)$ is determined from 21, the depression front velocity v may also be determined from relation

$$V=\frac{\mathrm{d}l}{\mathrm{d}t}=\frac{n}{1+n}\eta_1 t^{-1(1+n)} \tag{26}$$

where η_1 is obtained from 24, and V is the depression front velocity.

The integral 25 in Equation 22 cannot be evaluated analytically but may easily be evaluated numerically using for this purpose a standard quadrature procedure. Figure 1 shows the results obtained by means of the numerical integration from which the values of $J_n(1)=L_n$, i.e., at $\eta/\eta_1=1$ may be determined.

The solutions given by relations 22 and 23, expressing the pressure and velocity distributions in the flow system, have the merit of simplicity and the numerical results may easily be known from Figure 1.

At $x=0$ relation 23 gives

$$v_0(t)=\left(\frac{k}{\mu_{\mathrm{eff}}}\right)^{1/n}B^{1/(1-n)}\eta_1^{2/(1-n)}t^{-1/(1+n)} \tag{27}$$

so that, from a comparison with 18, it turns out

$$A=\left(\frac{k}{\mu_{\mathrm{eff}}}\right)^{1/n}B^{1/(1-n)}\eta_1^{2/(1-n)} \tag{28}$$

in which B and η_1 are determined from 24.

In order to express η_1 in terms of $df/d\eta|_{\eta=0}$, one can use the following approach. Assuming an average pressure in the zone $0<x<l(t)$, i.e., an average density, then taking into account 18, one can write

$$G_0(t)=\bar{\gamma}F\int_0^t v_0(\tau)\,\mathrm{d}\tau=\frac{\bar{\gamma}F(1+n)A}{n}t^{n/(1+n)} \tag{29}$$

On the other hand, one can also write

$$G_0(t)=F\bar{\gamma}l(t) \tag{30}$$

and consequently from 29 and 30 we obtain an expression for the front location in function of time:

$$l(t)=\frac{(1+n)A}{n}t^{n/(1+n)} \tag{31}$$

From relations 16 and 31, η_1 may be finally related to $\mathrm{d}f/\mathrm{d}\eta|_{\eta=0}$ by the expression

$$\eta_1=\frac{1+n}{n}\left(\frac{kp_w}{\mu_{\mathrm{eff}}}\right)^{1/n}\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)^{1/n}_{\eta=0} \tag{32}$$

or

$$\eta_1=\frac{1+n}{n}A=\text{constant} \tag{33}$$

Obviously, 32 represents an approximate relation derived from an approach in which the compressibility effect in the zone of elastic decompression, i.e., $0<x<l(t)$, has been ignored. To consider this effect, we should use instead of 30,

$$G_0(t)=gF\int_0^{l(t)}[\rho(x,t)-\rho_k]\,dx \tag{34}$$

For a slightly compressible fluid, the specific mass is related to pressure

$$\rho=\rho_k e^{\beta(p-p_k)}\cong\rho_k[1+\beta(p-p_k)] \tag{35}$$

where p_k is a reference pressure for which $\rho=\rho_k$. Equation 35 was also used in deriving the flow equations 1 and 2.

Introducing 35 into 34, we have

$$G_0(t)=g\rho_k\beta F\int_0^{l(t)}[p(x,t)-p_k]\,dx \tag{36}$$

This relation clearly shows that the determination of η_1 requires knowledge of pressure distribution $p(x,t)$ in the zone $0<x<l(t)$, which is the solution of Equation 7. As already mentioned, this solution requires for $n<1$ a formulation of a moving boundary problem, as presented and discussed in Pascal and Pascal.[2] As one can see from the results shown above, the theoretical evidence to support the existence of a decompression front, due to the elastic decompression of a slightly compressible fluid of pseudoplastic type and its location at any time, is physically demonstrated. With no loss of generality, we will illustrate further on the existence of self-similar solutions for the case of a dilatant fluid, i.e., $n>1$. This case may be analyzed without having to consider the formulation of a moving boundary problem, as required for $n<1$, i.e., a pseudoplastic fluid. For example, for $n=2$, Equations 12 and 13 become

$$\Phi(\eta)=-\left(\frac{kp_w}{\mu_{\text{eff}}}\right)^{1/2}\frac{t^{-2/3}}{v_0(t)}\left(\frac{df}{d\eta}\right)^{1/2} \tag{37}$$

and

$$\frac{d\Phi}{d\eta}=+\frac{\beta\phi p_w}{3}\frac{\eta t^{-2/3}}{v_0(t)}\frac{df}{d\eta} \tag{38}$$

where $v_0(t)$ is, according to the relation 18, $v_0(t)=At^{-2/3}$.

Equations 37 and 38 lead to

$$\frac{d\Phi}{d\eta}-b^2\eta\Phi^2=0;\qquad b^2=\frac{\beta\phi\mu_{\text{eff}}A}{3k} \tag{39}$$

From 20 when $n=2$ we have

$$f(\eta)=1+\int_0^{\eta}\frac{d\eta}{(C_1+a^2\eta^2/3)^2},\qquad 0<\eta<\infty \tag{40}$$

Therefore, the functions $f(\eta)$ and $\Phi(\eta)$, satisfying the boundary conditions specified in 14, can now be obtained from 39 and 40 and expressed as

$$f(\eta)=\frac{p(\eta)}{p_w}=1+\frac{\eta}{2C_1(C_1+a^2\eta^2/3)}+\frac{1}{2C_1\sqrt{C_1a^2/3}}\operatorname{arctg}\left(\sqrt{\frac{a^2}{3C_1}}\,\eta\right) \tag{41}$$

and

$$\Phi(\eta)=\frac{v(n)}{v_0(t)}=\frac{1}{1+b^2\eta^2/2} \tag{42}$$

with C_1 determined from condition $f(\infty)=p_k/p_w$

$$C_1=\left[\frac{\pi\sqrt{3}}{4a}\frac{p_w}{p_k-p_w}\right]^{2/3} \tag{43}$$

From 42 the velocity distribution will be

$$v(x,t)=\frac{At^{-2/3}}{1+\frac{b^2}{2}x^2t^{-4/3}} \tag{44}$$

Naturally, we now ask if the self-similar solutions of Equations 1 and 2 could also be found for the case of a constant velocity at the outface flow. Specifically, we are interested in knowing the pressure variation in time at the outface flow when a constant velocity is imposed there. In this situation functions $p(\eta)$ and $v(\eta)$ should be related to $f(\eta)$ and $\Phi(\eta)$ by the relations

$$p=p_k+(p_w(t)-p_k)f(\eta) \tag{45}$$

$$v=v_0\Phi(\eta);\qquad v_0=\text{constant} \tag{46}$$

As a result, the following boundary conditions arise

$$\eta=0\qquad f(0)=1\quad\text{and}\quad\Phi(0)=1 \tag{47}$$

$$\eta=\infty\qquad f(\infty)=0\quad\text{and}\quad\Phi(\infty)=0 \tag{48}$$

It is straightforward to show that Equations 1 and 2 become

$$\Phi^n=-\frac{k(p_w(t)-p_k)t^{-n(1+n)}}{\mu_{\text{eff}}v_0^n}\left(\frac{df}{d\eta}\right) \tag{49}$$

and

$$\frac{d\Phi}{d\eta}=+\frac{n\beta\phi(p_w(t)-p_k)\eta t^{-1/(1+n)}}{(1+n)v_0}\times\left[\frac{df}{d\eta}-\frac{1+n}{n}\frac{t}{\eta}f(\eta)\frac{d(\ln(p_w(t)-p_k))}{dt}\right] \tag{50}$$

From 49 and 50 it is evident that an analytical expression for the function $p_w(t)$ does not exist, such that these equations can be expressed in terms of variable η only. Consequently, the case corresponding to the boundary conditions 47 and 48 is not self-similar.

Plane radial flow

In this section we are concerned with the case of plane radial flow. Of particular interest, as we will see immediately, is the sensitivity of this case to the coupled effects specifically associated with the flow geometry and non-Newtonian behavior. For example, assuming a radial steady flow of an incompressible fluid then we have

$$v=\frac{Q_w}{2\pi hR};\qquad Q_w=\text{constant} \tag{51}$$

On the other hand, for a power-law fluid the apparent viscosity μ_{ap} is given by the relation

$$\mu_{ap}=H(\dot\gamma)^{n-1} \tag{52}$$

where $\dot\gamma$ is expressed in terms of flow velocity in porous medium

$$\dot\gamma=\frac{3n+1}{n}\frac{v}{\sqrt{8k\phi}} \tag{53}$$

Previous relations 51, 52, and 53 reveal that

$$\mu=\mu_w\left(\frac{R}{R_w}\right)^{1-n} \tag{54}$$

in which μ_w is the viscosity corresponding to R_w, i.e., at the well.

It is evident from 51 that for a power-law fluid of pseudoplastic behavior, i.e., $n<1$, the apparent viscosity is a monotonic increasing function of increasing radial distance. As a result, the rheological effects of non-Newtonian behavior on the radial flow in a porous medium will become more significant with increasing radial distance. This relevant result appears naturally as a consequence of coupled effects depending on the flow geometry and due to the fact that the apparent viscosity of a non-Newtonian fluid is flow rate dependent. As a result, the case of radial flow appears to be of great interest in oil reservoir engineering, showing implications of relation 54 on the flow behavior. Specifically, we have here a situation arising from relation 54, in which an unsteady flow of a slightly compressible fluid having a variable viscosity with radial distance is involved.

For a plane radial flow, Equations 1 and 2 become

$$-\frac{\partial p}{\partial R}=\frac{\mu_{\text{eff}}}{k}v^n \tag{55}$$

$$-\beta\phi\frac{\partial p}{\partial t}=\frac{\partial v}{\partial R}+\frac{v}{R} \tag{56}$$

These equations lead to

$$\left(\frac{\partial p}{\partial R}\right)^{(1-n)/n}\left[\frac{\partial^2 p}{\partial R^2}+\frac{n}{R}\frac{\partial p}{\partial R}\right]=na^2\frac{\partial p}{\partial t} \tag{57}$$

It is convenient to introduce the transformation

$$\eta=Rt^{-n/(1+n)} \tag{58}$$

and functions $f(\eta)$ and $\Phi(\eta)$ defined as

$$p=p_w f(\eta)$$
$$v=v_0(t)\Phi(\eta) \tag{59}$$

Using relations 58 and 59, Equations 55 and 56 may be written as

$$\Phi(\eta)=-\left(\frac{kp_w}{\mu_{\text{eff}}}\right)^{1/n}\frac{t^{-1/(1+n)}}{v_0(t)}\left(\frac{df}{d\eta}\right)^{1/n} \tag{60}$$

and

$$\frac{d\Phi}{d\eta}+\frac{\Phi}{\eta}=\frac{n\beta p_w\phi}{(1+n)v_0(t)}\eta t^{-1/(1+n)}\frac{df}{d\eta} \tag{61}$$

On the other hand, from Equations 57, 58, and 59, we have

$$\frac{d^2f}{d\eta^2}+\frac{n}{\eta}\frac{df}{d\eta}+\frac{na^2}{1+n}\eta\left(\frac{df}{d\eta}\right)^{(2n-1)/n}=0 \tag{62}$$

The boundary conditions associated with the above equations are

$$f(\eta_w)=1 \quad \text{and} \quad f(\infty)=\frac{p_k}{p_w} \tag{63}$$

$$\Phi(\eta_w)=1 \quad \text{and} \quad \Phi(\infty)=0$$

in which η_w is determined from 58 and expressed as

$$\eta_w=R_w t^{-n/(1+n)} \tag{64}$$

R_w being the well radius.

Following the same approach as in the one-dimensional case, we will derive the conditions for which the self-similar solutions for Equations 55 and 56 exist. For example, from 63 one has $\Phi(\eta_w)=1$, so from 60 one obtains

$$v_0(t)=-\left(\frac{kp_w}{\mu_{\text{eff}}}\right)^{1/n}t^{-1/(1+n)}A(t) \tag{65}$$

where

$$A(t)=\left(\frac{df}{d\eta}\right)^{1/n}_{\eta=\eta_w} \tag{66}$$

In determining the function $A(t)$ we are again faced with an identical problem as in the one-dimensional case. However, as we will see further on, the radial case is more difficult because A, taking into account 64, is no longer a constant as in the one-dimensional flow. Considering $n<1$, the determination of $(df/d\eta)^{1/n}_{\eta=\eta_w}$ also requires an approach based on the formulation of a moving boundary problem. As shown in Pascal and Pascal,[2] using this approach Equation 62 is reduced to

$$\frac{df}{d\eta}=\frac{1}{\eta^n}\left[\frac{(1-n)a^2}{(1+n)(3-n)}\right]^{n/(1-n)}(\eta_1^{3-n}-\eta^{3-n})^{n/(1-n)} \tag{67}$$

subject to the boundary conditions 63.

Taking into account that η_w^{3-n} may be neglected as compared with η_1^{3-n}, then from 67 it turns out

$$A(t)=\left(\frac{df}{d\eta}\right)^{1/n}_{\eta=\eta_w}=\frac{\eta_1^{(3-n)/(1+n)}}{R_w}\left[\frac{(1-n)a^2}{(1+n)(3-n)}\right]^{1/(1-n)}t^{n/(1+n)} \tag{68}$$

Once $A(t)$ is determined from 68, relation 65 may now be expressed as

$$v_0(t)=Bt^{(n-1)/(n+1)} \tag{69}$$

which shows that maintaining a constant pressure at the outface flow, the velocity must decline in time according to relation 69. B is a constant obtained from 65 and 68, while the relations determining η_1 are available in Pascal and Pascal[2] and therefore are not repeated here. However, we give here the expression for the pressure distribution

$$p(\eta)=p_k-\bar{B}\eta_1^{(1+n)/(1-n)}J_n\left(\frac{\eta}{\eta_1}\right), \qquad \eta_w<\eta<\eta_1 \tag{70}$$

where

$$J_n\left(\frac{\eta}{\eta_1}\right)=\int_{\eta/\eta_1}^{1}\xi^{-n}(1-\xi^{3-n})^{n/(1-n)}\,d\xi \tag{71}$$

The integral 71 has been numerically performed and its values are presented in Figure 2. Once η_1 is determined, the front location $l(t)$ at a given time may be predicted by means of relation 58 and expressed as

$$l(t)=\eta_1 t^{n/(1+n)} \tag{72}$$

As previous results indicate, the case when the well is producing at a constant pressure allows the self-similar solutions for Equations 55 and 56, provided that the fluid velocity $v_0(t)$ will decline there according to the relation 69. It is straightforward to show that these solutions no longer exist for a constant flow rate of production.

Self-similar solutions of the equations governing unsteady gas flow

In previous sections we have shown the existence of self-similar solutions of the equations governing unsteady flow of non-Newtonian fluids for a class of fluid mechanics problems of practical interest in oil reservoir engineering. Another class of problems arising currently in practice is related to the unsteady gas flow through a porous medium. While the non-Newtonian fluids of power law behavior, investigated previously, may be considered slightly compressible fluids, in which case the

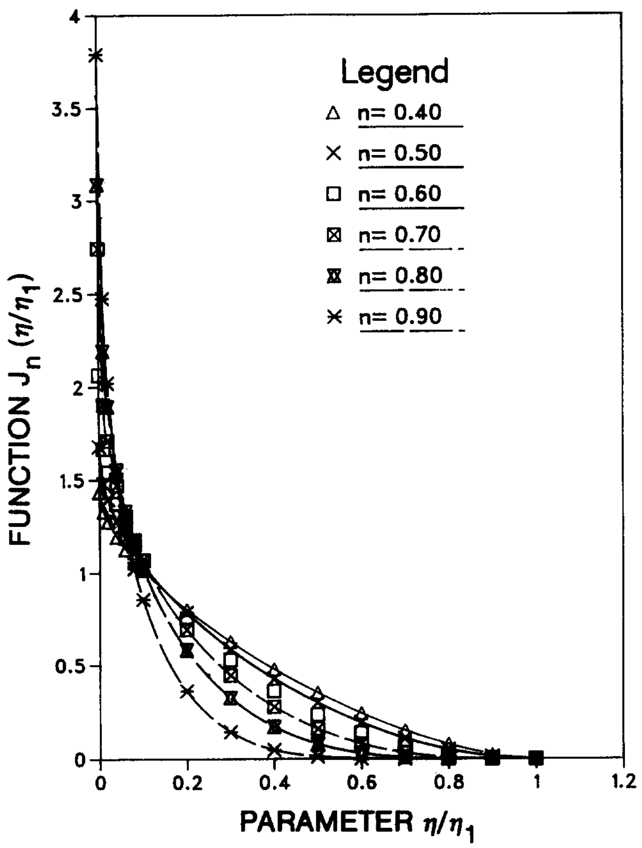

Figure 2 Rheological effect on the function J_n (η/η_1) for various n values, radial flow

relation 35 is valid, in the case of gas flow under high pressure this assumption is no longer valid.

Assuming an isothermal flow, i.e., at a constant temperature, instead of 35, one has

$$\rho=bp; \qquad b=\frac{1}{\bar{Z}RT} \tag{73}$$

in which $\bar{Z}$ is the compressibility factor expressed at average pressure.

The violation of Darcy's law for a relatively high velocity, particularly in the case of gas flow, has been reported in the literature for a long time. As the gas velocity in the pore space increases above a certain value, the linear relationship between the pressure gradient and velocity, inherent in Darcy's law, is no longer valid. The observed departure from Darcy's law was found to be directly related to the significance of the effects of convective accelerations and decelerations of the gas through varying cross-sectional areas in the pore space. Specifically, the nonlinear inertial term $v(\partial v/\partial x)$ was found to be directly responsible for the deviation from Darcy's law. As is well known, for a low velocity or a low Reynold's number, the viscous forces are significant, while for high velocity the inertial forces will dominate.

To include the inertial effects associated with the nonlinear term $v(\partial v/\partial x)$ in the motion equation, an appropriate relationship between the pressure gradient and velocity should be used. Such a relation, experimentally found and theoretically justified, is expressed as

$$-\frac{\partial p}{\partial x}=\frac{\mu}{k}v+\bar{\beta}\rho v^2 \tag{74}$$

where ρ is the specific mass; i.e., $\rho=\gamma/g$, γ being the density; and $\bar{\beta}$ is an empirical coefficient or fitting parameter, termed as inertial flow resistance coefficient. The numerical values of $\bar{\beta}$ in terms of permeability and porosity are available in the literature.

It is convenient to introduce the notation $q=\rho v$, termed mass velocity, in which case from 73 and 74, one obtains

$$-\frac{b}{2}\frac{\partial p^2}{\partial x}=\frac{\mu}{k}q+\bar{\beta}q^2 \tag{75}$$

Experimental observations show, however, that at high velocity, occurring in some problems of practical interest in the gas industry, the linear term $(\mu/k)q$ related to the Darcy flow may be in general neglected in 75. In this section we are particularly interested in non-Darcy flow which is adequately described by the equations

$$-\frac{b}{2}\frac{\partial p^2}{\partial x}=\bar{\beta}q^2 \tag{76}$$

and

$$-\frac{\partial q}{\partial x}=b\phi\frac{\partial p}{\partial t} \tag{77}$$

To make the problem as general as possible, we will write Equations 76 and 77

$$-\frac{\partial p^m}{\partial x}=\frac{2}{b}\bar{\beta}q^n \tag{78}$$

and

$$-\frac{\partial q}{\partial x}=b\phi\frac{\partial p}{\partial t} \tag{79}$$

Power-law exponent n in 78 does not have the same meaning as in the case of non-Newtonian power law fluid. According to the experimental observations we may have $n=1$ or $n=2$, while m must be equal to 2.

For the particular case $m=2$ and $n=2$, we have the non-Darcy flow governed by Equations 76 and 77, while for $m=2$ and $n=1$ we have a Darcy flow in which $\bar{\beta}=\mu/k$. To show the existence of self-similar solutions for the system of nonlinear Equations 78 and 79, we will consider the case of constant mass velocity at the outface flow, using the expressions

$$\begin{aligned} &p=p_k+\Delta p(t)f(\eta); \qquad \Delta p(t)=p_w(t)-p_k \\ &q=q_0\Phi(\eta); \qquad q_0=\text{constant} \end{aligned} \tag{80}$$

in which the self-similar variable η for Equations 78 and 79 is now defined as

$$\eta=xt^{-m(1+m)} \tag{81}$$

rather than by relation 4.

Equations 80 and 81 allow us to express the system 78 and 79 in terms of unknown functions $f(\eta)$ and $\Phi(\eta)$

$$\Phi^n(\eta)=-\frac{2m\bar{\beta}t^{-m/(1+m)}}{bq_0^n}\Delta p(t)(p_k+\Delta pf)^{m-1}\left(\frac{\mathrm{d}f}{\mathrm{d}\eta}\right) \tag{82}$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\eta}=+\frac{mb\phi}{(1+m)q_0}\Delta p(t)\eta t^{-1/(1+m)} \times\left[\frac{\mathrm{d}f}{\mathrm{d}\eta}-\frac{m+1}{m}f(\eta)\frac{t}{\eta}\frac{1}{\Delta p(t)}\frac{\mathrm{d}\Delta p(t)}{\mathrm{d}t}\right] \tag{83}$$

Equations 82 and 83 are subject to the following boundary conditions:

$$\eta=0 \quad f(0)=1 \quad \text{and} \quad \Phi(0)=1$$
$$\eta=\infty \quad f(\infty)=0 \quad \text{and} \quad \Phi(\infty)=0 \tag{84}$$

Since $f(0)=1$, then from 83 we have at $\eta=0$

$$\frac{d\Delta p(t)}{dt}=\frac{q_0}{b\phi}At^{-m/(1+m)}; \qquad A=\left.\frac{d\Phi}{d\eta}\right|_{\eta=0} \tag{85}$$

where it turns out that $\Delta p(t)$ should be expressed as

$$\Delta p(t)=Ct^{1/(1+m)}; \qquad C=-\frac{(1+m)q_0}{6\phi}\left(\frac{d\Phi}{d\eta}\right)_{\eta=0}=\text{constant} \tag{86}$$

Considering 86, Equations 82 and 83 could be expressed in terms of variable η only, provided that the approximation

$$p_k+\Delta pf\cong\Delta pf \tag{87}$$

is satisfied. However, from 80 we have $0<f<1$, so that condition 87 could not be met in some situations of practical interest, except the case when the initial pressure, i.e., the pressure at $t=0$, is $p_k=0$. Naturally, this case is more appropriate to the situation when a fluid is injected into reservoir having $p_k\cong 0$.

Assuming that 87 is a valid approximation, then, taking into account 86, we may rewrite Equations 82 and 83 as follows:

$$\Phi^n(\eta)=-\frac{2m\bar{\beta}C}{bq_0^n}f^{m-1}\frac{df}{d\eta} \tag{88}$$

and

$$\frac{d\Phi}{d\eta}=\frac{mb\phi C}{1+m}\eta\left(\frac{df}{d\eta}-\frac{1}{m}\frac{f}{\eta}\right) \tag{89}$$

These equations lead to

$$\frac{d}{d\eta}\left[-\alpha f^{m-1}\frac{df}{d\eta}\right]^{1/n}=\frac{b\phi C}{1+m}\left(m\eta\frac{df}{d\eta}-f\right) \tag{90}$$

where

$$\alpha=\frac{2m\bar{\beta}C}{bq_0^n} \tag{91}$$

It is evident that the determining of $f(\eta)$ and $\Phi(\eta)$, satisfying the conditions specified in 84, will require the use of an appropriate numerical approach.

As already mentioned, the cases of special interest in some practical applications, which we will analyze further on, are $m=2$ and $n=2$, as well as $m=2$ and $n=1$.

For a non-Darcy flow, i.e., $m=2$ and $n=2$, Equations 88, 89, and 90 become

$$\Phi^2=-\alpha f\frac{df}{d\cdot} \tag{92}$$

$$\frac{d\Phi}{d\eta}=\frac{b\phi C}{3}\left(2\eta\frac{df}{d\eta}-f\right) \tag{93}$$

$$\tfrac{1}{3}b\phi C\left(2\eta\frac{df}{d\eta}-f\right)=\frac{d}{d\eta}\left[-\alpha f\frac{df}{d\eta}\right]^{1/2} \tag{94}$$

while for Darcy flow, i.e., $m=2$ and $n=1$, instead of 92 and 94, we have

$$\Phi=-\alpha f\frac{df}{d\eta} \tag{95}$$

and

$$\frac{1}{3}\frac{b\phi C}{\alpha}\left(2\eta\frac{df}{d\eta}-f\right)=-\frac{d}{d\eta}\left[f\frac{df}{d\eta}\right] \tag{96}$$

It is straightforward to show that the transformation

$$f(\eta)=\exp\left(-2\int_0^\eta u^2\,d\eta\right) \tag{97}$$

yields from 94 the Abell's equation

$$\frac{du}{d\eta}-2u^3+4\delta u^2\eta-\delta=0; \qquad \delta=\frac{b\phi C}{3\sqrt{2\alpha}} \tag{98}$$

which, obviously, requires a numerical integration to determine $f(\eta)$.

On the other hand, an approximate analytical solution for the system of nonlinear equations 88 and 89 could be determined by means of the perturbation method. In this case, the functions $f(\eta)$ and $\Phi(\eta)$ should be expressed as a power series expansion in terms of a small parameter ε.

Finally, we will now consider the situation when at the outface a constant pressure is imposed. We are particularly interested in knowing the qualitative behavior of the velocity in time at the outface flow, in order to maintain a constant pressure there. For this case the relations 10 and 11, as well as the boundary conditions specified in Equation 14, are valid. However, instead of transformation 81, used for a constant velocity, we must use for the case of constant pressure the following transformation

$$\eta=xt^{-n/(1+n)} \tag{99}$$

Introducing 99 into 78 and 79 and considering 10 and 11 one obtains

$$\Phi^n=\frac{bp_w^m t^{-n/(1+n)}}{2\bar{\beta}q_0^n(t)}\frac{df^m}{d\eta} \tag{100}$$

and

$$\frac{d\Phi}{d\eta}=\frac{nb\phi p_w}{(1+n)q_0(t)}\eta t^{-1(1+n)}\frac{df}{d\eta} \tag{101}$$

These equations indicate that a self-similar solution exists provided that

$$q_0(t)=Ct^{-1(1+n)}; \qquad C=\left[-\frac{2\bar{\beta}}{bp_w^m}\left.\frac{df^m}{d\eta}\right|_{\eta=0}\right]^{1/n} \tag{102}$$

An expected result is that Equation 102 is similar to 18, except for the coefficient C which in this case requires the knowledge of $df^m/d\eta|_{\eta=0}$, in which function $f(\eta)$ is the solution of the equation

$$\frac{d}{d\eta}\left[-\frac{df^m}{d\eta}\right]^{1/n}-b^2\eta\frac{df}{d\eta}=0; \qquad b^2=\frac{n\phi}{1+n}b^{(1-n)/n}(2\bar{\beta})^{1/n}p_w^{(n-m)/n} \tag{103}$$

obtained from Equations 100, 101, and 102.

Concluding remarks

In this investigation we have shown the conditions for which the self-similar solutions of the equations governing unsteady flow of non-Newtonian fluids of power-law behavior exist. For example, these solutions exist when at the outface flow we have a constant pressure, while for a constant flow rate imposed there the self-similar solutions no longer exist. The self-similar solutions corresponding to a constant pressure yield a qualitative analytical expression for the flow rate variation in time.

However, to determine completely the flow rate variation in time, we must integrate the nonlinear differential equation 15 to know $df/d\eta|_{\eta=0}$, as can be seen from 16.

Limitations of the self-similar solutions may appear in some cases of practical interest. For example, in the case of a constant flow rate at the outface flow for a non-Newtonian power law fluid a self-similar solution does not exist, while this case for non-Darcy gas flow allows a self-similar solution.

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